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REGULAR IMPLEMENTABILITY AND ITS APPLICATION TO STABILIZATION OF SYSTEM BEHAVIORS

M.N. Belur, H.L. Trentelman and J.C. Willems [1]

Abstract

In this paper we study control by interconnection of linear differential systems. We give necessary and sufficient conditions for regular implementability of a given linear differential system. We formulate the problems of stabilization and pole placement as problems of finding suitable, regularly implementable sub-behaviors of the manifest plant behavior. The problem formulations and their resolutions are completely representation free, and specified in terms of the system dynamics only. Control is viewed as regular interconnection. A controller is a system that constrains the plant behavior through a distinguished set of variables, namely, the control variables. The issue of implementation of a controller in the feedback configuration and its relation to regularity of interconnection is addressed. Freedom of disturbances in a plant and regular interconnection with a controller also turn out to be inter-related.

Keywords: Behaviors, regular implementability, stabilization, pole placement, interconnection, controller implementation.

1 Introduction and notation

In this paper we discuss the issue of stabilization of linear dynamical systems. The problem is studied in the behavioral context and control is viewed as interconnection. This view of treating control problems has been used before in, for example, [15], [7], [3] and [2], in an H_∞ control context in [4], [5], [1], [9], [10], [11], [12] and [13], for adaptive control in [8], and for distributed systems in [6]. In contrast to [15] where the problems of stabilization and pole placement were considered for the case that *all* system variables are available for interconnection (the so-called full information case), we work in the generality that we are allowed to use only *some* of the system variables for the purpose of interconnection. These variables are called the control variables. Restricting oneself to using only the control variables for interconnection introduces the issue of *implementability* into the control problem, see [12] and [8]. In the context of stabilization, an important role is played by the notion of *regular implementability.* We establish necessary and sufficient conditions for a given behavior to be regularly implementable (section 2). This result is then applied to solve the problems of stabilization and pole placement by interconnection (section 3). The general problem formulation reduces to some important special cases. Section 4 contains the case of filtering. Implementation of a controller in a feedback configuration plays a very prominent role in control theory. This issue is addressed in section 5. Finally, in section 6 we give a motivation for the fact that in our problem formulations we restrict ourselves to regular interconnections.

We first discuss some of the notation to be used in this paper, and review some basic facts from the behavioral approach. We use the standard notation $\mathbb{R}^{\mathtt{n}}$ for the n-dimensional real Euclidean space. Often, the notation $\mathbb{R}^{\mathtt{w}}$ is used if w denotes a typical element of that vector space, or a typical function taking its value in that vector space. The ring of (one-variable) polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$. $\mathbb{R}^{\mathtt{n}_1\times \mathtt{n}_2}[\xi]$ denotes the set of matrices with $\mathtt{n}_1$ rows and $\mathtt{n}_2$ columns in which each entry is an element of $\mathbb{R}[\xi]$. We use the notation $\mathbb{R}^{\bullet\times \mathtt{n}_2}$ when the number of rows is unspecified.

In this paper, we deal with linear time-invariant differential systems, in short, linear differential systems. A linear differential system is defined as a dynamical system whose behavior $\mathfrak{B}$ is equal to the set of solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a polynomial matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that

$$\mathfrak{B} = \{w \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0\}.$$

Here, $\mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ denotes the space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, and $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is understood to hold in the distributional sense. The set of linear differential systems with manifest variable w taking its value in $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{L}^{\mathtt{w}}$.

We make a clear distinction between the behavior as defined as the space of all solutions of a set of (differential) equations, and the set of equations itself. A set of equations in terms of which the behavior is defined, is called a *representation* of the behavior. Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ be a polynomial matrix. If a behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ then we call this a kernel representation of $\mathfrak{B}$. Further, a kernel representation is

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said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least g rows. A given kernel representation, $R(\frac{d}{dt})w = 0$, is minimal if and only if the polynomial matrix R has full row rank. We speak of a system as the behavior $\mathfrak{B}$, one of whose representations is given by $R(\frac{d}{dt})w = 0$ or just $Rw = 0$. The '$\frac{d}{dt}$' is often suppressed to enhance readability. We will also encounter behaviors $\mathfrak{B}$ with manifest variable w, that are represented by equations of the form $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$, in which an auxiliary, latent variable ℓ appears. Here, R and M are polynomial matrices with the same number of rows. Through such an equation, we can consider the subspace of all $w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ for which there exists an $\ell \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$ such that the equation holds. A technical detail is that, by itself, this subspace is not an element of $\mathfrak{L}^{\mathtt{w}}$, because it is not a closed subspace (closed in the topology of $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$). Therefore, we call $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ a latent variable representation of $\mathfrak{B}$ if

$$\mathfrak{B} = \{w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, \ell \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \text{ such that } R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell\}^{\mathrm{closure}}$$

where the closure is taken in the $\mathcal{L}_1^{\mathrm{loc}}$ topology. Then, by the elimination theorem (see [7], chapter 6, in particular, theorem 6.2.6), $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$.

The space of functions that are infinitely often differentiable with domain $\mathbb{R}$ and co-domain $\mathbb{R}^{\mathtt{w}}$, is denoted by $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be represented by the kernel representation $R(\frac{d}{dt})w = 0$ with $\mathrm{rank}(R) < \mathtt{w}$ (which also means that it is under-determined). Then some components of $w = (w_1, w_2, \ldots, w_{\mathtt{w}})$ are unconstrained by the requirement $w \in \mathfrak{B}$. These components are termed as ***inputs*** or are said to be ***free*** (in the $\mathfrak{C}^\infty$ sense, for the purpose of this paper). The maximum number of such components is called the ***input cardinality*** of $\mathfrak{B}$ (denoted as $\mathtt{m}(\mathfrak{B})$). Once $\mathtt{m}(\mathfrak{B})$ free components are chosen, the remaining $\mathtt{w} - \mathtt{m}(\mathfrak{B})$ components are determined up to a finite dimensional affine subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}-\mathtt{m}(\mathfrak{B})})$. These are called ***outputs***, and the number of outputs is denoted by $\mathtt{p}(\mathfrak{B})$. Thus, possibly after permutation of components, $w \in \mathfrak{B}$ can be partitioned as $w = (u, y)$, with the $\mathtt{m}(\mathfrak{B})$ components of u as inputs, and the $\mathtt{p}(\mathfrak{B})$ components of y as outputs. We say that (u, y) is an input/output partition of $w \in \mathfrak{B}$, with input u and output y. The input/output structure of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is reflected in its kernel representations as follows. Suppose $R(\frac{d}{dt})w = 0$ is a minimal kernel representation of $\mathfrak{B}$. Partition $R = [Q\ P]$, and accordingly $w = (w_1, w_2)$. Then $w = (w_1, w_2)$ is an i/o partition (with input w_1 and output w_2) if and only if P is square and nonsingular. In general, there exist many input/output partitions, but the integers $\mathtt{m}(\mathfrak{B})$ and $\mathtt{p}(\mathfrak{B})$ are invariants associated with a behavior. It can be verified that $\mathtt{p}(\mathfrak{B})$ is equal to the rank of the polynomial matrix in any (not necessarily minimal) kernel representation of $\mathfrak{B}$ (for details see [7]).

A behavior whose input cardinality is equal to 0 is called ***autonomous***. An autonomous behavior $\mathfrak{B}$ is said to be stable, if for all $w \in \mathfrak{B}$ we have $w(t) \to 0$ as $t \to \infty$. In the context of stability, we often need to describe regions of the complex plane $\mathbb{C}$. We denote the closed right-half of the complex plane by $\mathbb{C}^+$ and the open left-half complex plane by $\mathbb{C}^-$. A polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ is called *Hurwitz* if $\mathrm{rank}(R(\lambda)) = \mathtt{w}$ for all $\lambda \in \mathbb{C}^+$. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is represented by $R(\frac{d}{dt})w = 0$ then $\mathfrak{B}$ is stable if and only if R is Hurwitz.

For autonomous behaviors, we also speak about poles of the behavior. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be autonomous. Then there exists an $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ such that $\mathfrak{B}$ is represented minimally by $R(\frac{d}{dt})w = 0$. We can choose R such that $\det(R)$ is a monic polynomial. This monic polynomial is denoted by $\chi_{\mathfrak{B}}$ and is called *the characteristic polynomial of* $\mathfrak{B}$. It can be shown that $\chi_{\mathfrak{B}}$ depends only on $\mathfrak{B}$, and not on the polynomial matrix R we used to define $\chi_{\mathfrak{B}}$. The *poles* of $\mathfrak{B}$ are defined as the roots of $\chi_{\mathfrak{B}}$. Note that $\chi_{\mathfrak{B}} = 1$ if and only if $\mathfrak{B} = 0$. A behavior is stable if and only if all its poles are in $\mathbb{C}^-$.

Finally, we review the concept of controllability in the context of the behavioral approach. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is *controllable* if for all $w_1, w_2 \in \mathfrak{B}$, there exists a $T \geq 0$ and a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t + T) = w_2(t)$ for $t \geq 0$. A weaker notion is ***stabilizability***, which is defined as follows. A behavior $\mathfrak{B}$ is stabilizable if for all $w_1 \in \mathfrak{B}$, there exists a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) \to 0$ as $t \to \infty$. Thus every trajectory in a stabilizable behavior $\mathfrak{B}$, can be steered to 0, asymptotically.

Often, we encounter behaviors $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ that are neither autonomous nor controllable. The ***controllable part*** of a behavior $\mathfrak{B}$ is defined as the largest controllable sub-behavior of $\mathfrak{B}$. This is denoted by $\mathfrak{B}_{\mathrm{cont}}$. A given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ can always be decomposed as $\mathfrak{B} = \mathfrak{B}_{\mathrm{cont}} \oplus \mathfrak{B}_{\mathrm{aut}}$, where $\mathfrak{B}_{\mathrm{cont}}$ is the (unique) controllable part of $\mathfrak{B}$, and $\mathfrak{B}_{\mathrm{aut}}$ is a (non-unique) autonomous sub-behavior of $\mathfrak{B}$. For details we refer to [7].

We also deal with systems in which the signal space comes as a product space, with the first component viewed as an observed, and the second as a to-be-deduced variable. We talk about observability (in such systems). Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with manifest variable $w = (w_1, w_2)$, w_2 is said to be ***observable*** from w_1 if $(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$. Let $R_1(\frac{d}{dt})w_1 + R_2(\frac{d}{dt})w_2 = 0$ be a kernel representation of $\mathfrak{B}$. Then observability of w_2 from w_1 is equivalent to $R_2(\lambda)$ having full column rank for all $\lambda \in \mathbb{C}$. The weaker notion of ***detectability*** is defined along similar lines. Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$, w_2 is said to be detectable from w_1 if $(w_1, w_2'),(w_1, w_2'') \in \mathfrak{B}$ implies $w_2'(t) - w_2''(t) \to 0$ as $t \to \infty$. In the above kernel representation, detectability of w_2 from w_1 is equivalent to $R_2(\lambda)$ having full column rank for all $\lambda \in \mathbb{C}^+$. For details, see [7].

2 Regular implementability

Suppose we have a plant to be controlled, with two types of variables. In the given plant, the variables whose trajectories we intend to shape (called the *to-be-controlled variables*), are denoted by w. These to-be-controlled variables can be controlled through a set of *control variables* c, over which we can "attach" a controller. These are the variables, that can be measured and/or actuated upon. Often we have some common components in w and c. We formulate the problem, however, for the general case, in which we have access to just the control variables c.

Before the controller acts, there are two behaviors of the plant that are relevant: $\mathcal{P}_{\text{full}}$ (called the *full plant behavior*) that formalizes the dynamics of the variables w and c, and the behavior $\mathcal{P}$ (called the *manifest plant behavior*) that formalizes the dynamics of the to-be-controlled variables w only. Thus

$$\begin{aligned}\mathcal{P}_{\text{full}} = \{(w,c) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}+\mathtt{c}}) \mid (w,c) \\ \text{satisfies the plant equations }\},\\ \mathcal{P} = \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists\, c \text{ such that} \\ (w,c) \in \mathcal{P}_{\text{full}}\}^{\text{closure}}.\end{aligned}$$

In this paper, we assume that the plant is a linear differential system, i.e. $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. The particular representation by which it is given, is immaterial to us. The manifest plant behavior $\mathcal{P}$ is obtained by *eliminating* c from $\mathcal{P}_{\text{full}}$, so, by the elimination theorem, $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$.

A controller restricts the trajectories that c can assume and is described by a *controller behavior* $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$:

$$\mathcal{C} = \{c \in \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{c}}) \mid c \text{ satisfies the controller equations}\}.$$

The *full controlled behavior* $\mathcal{K}_{\text{full}}$ is obtained by taking the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through the variable c and is defined as:

$$\mathcal{K}_{\text{full}} = \{(w,c) \mid (w,c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}.$$

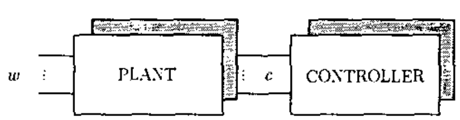

Figure 1: The plant and controller after interconnection

The *manifest controlled behavior* $\mathcal{K}$ is obtained from $\mathcal{K}_{\text{full}}$ by eliminating c and is defined as:

$$\mathcal{K} = \{w \mid \exists\, c \in \mathcal{C} \text{ such that } (w,c) \in \mathcal{P}_{\text{full}}\}^{\text{closure}}. \quad (1)$$

In that case we say that $\mathcal{K}$ is implemented by $\mathcal{C}$, or $\mathcal{C}$ implements $\mathcal{K}$ through c. A given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *implementable with respect to* $\mathcal{P}_{\text{full}}$ *by interconnection through* c, if there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, such that $\mathcal{K}$ is implemented by $\mathcal{C}$. If it is clear from the context, we often suppress the specifications 'w.r.t. $\mathcal{P}_{\text{full}}$' and 'through c'. An important issue is the question which $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ are implementable, i.e. for which $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that (1) holds. A crucial concept to answer this question is the notion of hidden behavior: the *hidden behavior* $\mathcal{N}$ is the behavior consisting of the plant trajectories that occur when the control variables are zero:

$$\mathcal{N} = \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid (w,0) \in \mathcal{P}_{\text{full}}\}.$$

We have access to only the control variables c - hence the notion of $\mathcal{N}$ being hidden from the control variables.

The following proposition from [12] settles the question of implementability for a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. We refer to this proposition as the controller implementability theorem.

Proposition 1 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a given full plant behavior, and let $\mathcal{P},\mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$ be the manifest plant behavior and hidden behavior, respectively. Then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by interconnection through c if and only if*

$$\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}.$$

In addition to implementability issues, the hidden behavior $\mathcal{N}$ plays a role in observability and detectability of $\mathcal{P}_{\text{full}}$. It can be easily seen that, in $\mathcal{P}_{\text{full}}$, w is observable from c if and only if $\mathcal{N} = 0$, and w is detectable from c if and only if $\mathcal{N}$ is stable.

Roughly speaking, for a given $\mathcal{P}_{\text{full}}$ we want to find a controller $\mathcal{C}$ such that the manifest controlled behavior $\mathcal{K}$ has desired properties. However, we shall restrict ourselves to $\mathcal{C}$'s such that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. A motivation for this is provided in section 6. The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is called *regular* if

$$\mathtt{p}(\mathcal{K}_{\text{full}}) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C}),$$

i.e., if the output cardinalities of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ add up to that of $\mathcal{K}_{\text{full}}$.

A given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *regularly implementable* if there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $\mathcal{K}$ is implemented by $\mathcal{C}$, and if the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. Similar to plain implementability, an important question is under what conditions a given sub-behavior $\mathcal{K}$ of $\mathcal{P}$ is regularly implementable. The following theorem is the main result of this section, and provides necessary and sufficient conditions for this:

Theorem 2 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $\mathcal{P},\mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$ be the corresponding manifest plant behavior and hidden behavior respectively. Let $\mathcal{P}_{\text{cont}}$ be the controllable part of $\mathcal{P}$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then, $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$*

by regular interconnection through c if and only if the following conditions are satisfied:

$$\boxed{\begin{array}{c} \mathcal{N} \subset \mathcal{K} \subset \mathcal{P} \\ \mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P} \end{array}}$$

The above theorem has two conditions. The first one is exactly the condition for implementability through c (as in the controller implementability theorem). The second condition formalizes the notion that the autonomous part of $\mathcal{P}$ is taken care of by $\mathcal{K}$. While the autonomous part of $\mathcal{P}$ is not unique, $\mathcal{P}_{\text{cont}}$ is. This makes verifying the regular implementability of a given $\mathcal{K}$ computable. As a consequence of this theorem, note that if $\mathcal{P}$ is controllable, then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable if and only if it is implementable.

3 Pole placement and stabilization

In this section we discuss the problems of pole placement and stabilization. The problem statements and the theorems involve the behaviors of the plant, etc. which have been defined in the previous section.

Pole placement problem : Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, find conditions under which there exists, and compute, for every monic $r \in \mathbb{R}[\xi]$ a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that:

- the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular,
- the manifest controlled behavior $\mathcal{K}$ has characteristic polynomial r.

Suppressing the controller $\mathcal{C}$ from the problem formulation, the problem can alternatively be stated as:

> Given $\mathcal{P}_{\text{full}}$, find conditions under which there exists, and compute, for every monic $r \in \mathbb{R}[\xi]$ a regularly implementable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\chi_{\mathcal{K}} = r$.

When the manifest controlled behavior $\mathcal{K}$ is only required to be stable, we refer to the problem as that of stabilization.

Stabilization problem : Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, find conditions for the existence of, and compute $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that

- the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular,
- the manifest controlled behavior $\mathcal{K}$ is stable.

Again, suppressing the controller $\mathcal{C}$ from the formulation, the stabilization problem can be restated as:

> Given $\mathcal{P}_{\text{full}}$, find conditions for the existence of, and compute a behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ that is stable and regularly implementable.

The main results of this section are the following theorems, which establish necessary and sufficient conditions for pole placement and stabilization.

> **Theorem 3 :** *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. For every monic $r \in \mathbb{R}[\xi]$, there exists a regularly implementable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\chi_{\mathcal{K}} = r$ if and only if $\mathcal{N} = 0$ and $\mathcal{P}$ is controllable, equivalently, if and only if:*
>
> - *in $\mathcal{P}_{\text{full}}$, w is observable from c,*
> - *$\mathcal{P}$ is controllable.*

> **Theorem 4 :** *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. There exists a regularly implementable stable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ if and only if $\mathcal{N}$ is stable and $\mathcal{P}$ is stabilizable, equivalently, if and only if:*
>
> - *in $\mathcal{P}_{\text{full}}$, w is detectable from c,*
> - *$\mathcal{P}$ is stabilizable.*

Note that, neither in the problem formulations nor in the conditions appearing in theorems 3 and 4, do representations of the given plant appear. Indeed, our problem formulations and their resolutions are completely *representation free*, and are formulated purely in terms of properties of the *behavior* $\mathcal{P}_{\text{full}}$. Thus, our treatment of the pole placement and stabilization problems is genuinely behavioral. Of course, theorems 3 and 4 are applicable to any particular representation of $\mathcal{P}_{\text{full}}$ as well.

In both the stabilization problem and the pole placement problem, we have restricted ourselves to regular interconnections. We give an explanation for this in section 6. At this point we note that if in the above problem formulations we replace "regularly implementable" by merely "implementable", then in the stabilization problem a necessary and sufficient condition for the existence of $\mathcal{K}$ is that $\mathcal{N}$ is stable (equivalently: in $\mathcal{P}_{\text{full}}$, w is detectable from c). In the pole placement problem, necessary and sufficient conditions are that $\mathcal{N} = 0$ (i.e., in $\mathcal{P}_{\text{full}}$, w is observable from c) and that $\mathcal{P}$ is not autonomous.

4 The filtering problem

Our general problem formulation of finding a regularly implementable, stable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, for a given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, includes also a problem that is, strictly speaking, not a control problem, but rather a *filtering problem.*

Consider the set-up of figure 2. The *observed plant* $\mathcal{P}_{\text{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ has two types of variables, w and y. w is a variable that we want to *estimate* and y is a variable that we *measure.*

A *filter* is a system $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$, with variables $(y, \hat{w})$. The idea is to find a filter $\mathcal{F}$ such that in the interconnection of $\mathcal{P}_{\text{obs}}$ and $\mathcal{F}$ through y (the measured variable), $\hat{w}$

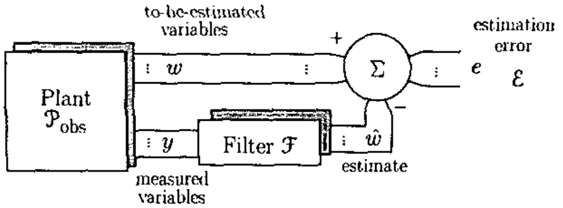

Figure 2: Plant and observer configuration

becomes an estimate of w. In order to formalize this, for a given filter $\mathcal{F}$ we define the associated *estimation error behavior* $\mathcal{E}$ by

$$\begin{aligned}\mathcal{E} = \{&e \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, w, \hat{w}, y \text{ such that:} \\ &(w, y) \in \mathcal{P}_{\text{obs}},\ (y, \hat{w}) \in \mathcal{F} \text{ and } e = w - \hat{w}\}^{\text{closure}}\ . \quad (2)\end{aligned}$$

Here, as before, the closure is taken with respect to the topology of $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. If $\mathcal{E}$, $\mathcal{P}_{\text{obs}}$ and $\mathcal{F}$ are related via equation (2), we say that $\mathcal{E}$ is *implemented by the filter* $\mathcal{F}$. Given $\mathcal{P}_{\text{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$, a given behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{w}}$ is called *implementable* (with respect to $\mathcal{P}_{\text{obs}}$) if there exists a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ such that $\mathcal{E}$ is implemented by $\mathcal{F}$. The question what $\mathcal{E}$'s are implementable is answered in the following lemma. In the following, let $\mathcal{N}$ be the hidden behavior associated with $\mathcal{P}_{\text{obs}}$, i.e.,

$$\mathcal{N} = \{w \mid (w, 0) \in \mathcal{P}_{\text{obs}}\}.$$

Lemma 5 : *Let $\mathcal{P}_{\text{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$. Then we have:*

1. *The behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{w}}$ is implementable if and only if $\mathcal{N} \subset \mathcal{E}$.*
2. *If $\mathcal{E}$ is autonomous and implementable, it can be implemented by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ such that, in $\mathcal{F}$, y is input and $\hat{w}$ output.*

The problem we want to consider in this section is to find a filter that makes the estimation error behavior *stable*. The following theorem states when such a filter exists.

Theorem 6 : *Let $\mathcal{P}_{\text{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$. There exists a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ such that the estimation error $\mathcal{E}$ is stable if and only if, in $\mathcal{P}_{\text{obs}}$, w is detectable from y. In that case, there exists a filter such that the measured variable y is input and the estimate $\hat{w}$ is output.*

5 Input/output partition

In the classical view of control, a controller is, in general, considered to be a *feedback processor* that generates control inputs for the plant on the basis of measured outputs of the plant. In our set-up, controller behaviors are obtained directly from the full plant. It is important to know a priori when such controlled behavior is implementable by a feedback processor. Results on this have been obtained in [15], [11] and [12]. We extend these results here for the problems considered in this paper.

Our first result states that if $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable and autonomous (so, in particular, if it is stable or has prescribed characteristic polynomial), then for any controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$ there exists a partition of the control variable c such that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is, in fact, a feedback interconnection:

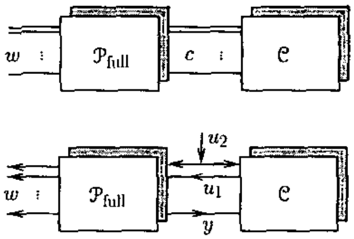

Figure 3: Feedback interconnection of $\mathcal{P}$ and $\mathcal{C}$

Theorem 7 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be autonomous and regularly implementable through c, and let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be a controller that regularly implements $\mathcal{K}$. Then, possibly after permuting its components, there exists a partition of c into $c = (y, u_1, u_2)$ such that:*

- *for $(w, y, u_1, u_2) \in \mathcal{P}_{\text{full}}$, (u_1, u_2) is input and (w, y) is output,*
- *for $(y, u_1, u_2) \in \mathcal{C}$, (y, u_2) is input and u_1 is output,*
- *for $(w, y, u_1, u_2) \in \mathcal{K}_{\text{full}}$, u_2 is input and (w, y, u_1) is output.*

As a special case, when $\mathcal{K}_{\text{full}}$ is autonomous, we interpret u_2 as having zero components. Figure 3 depicts how the control variables are partitioned into inputs and outputs in order to implement the controller behavior in a feedback configuration.

The above theorem *assigns* an input/output partition without modifying the controller itself. Often, we are not allowed to choose an input/output partition, because we are given *a priori* that some variables are sensors, while others are actuators. Hence, necessarily, the sensors are plant outputs and should, correspondingly, be controller inputs. The actuators, then, are inputs to the plant. In the following theorem we show that if our plant $\mathcal{P}_{\text{full}}$ has an a priori given input/output structure with respect to sensors and actuators, and if $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable and autonomous, then $\mathcal{K}$ can be regularly implemented by a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that takes the sensors as input, and actuates part of the plant actuators. Since $\mathcal{K}_{\text{full}}$ is again not necessarily autonomous, some control variables remain free. These can be interpreted as plant actuators which are not being used for the control of the to-be-controlled variables.

Theorem 8 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{y}+\mathtt{u}}$ with to-be-controlled variable w and control variable $c = (y, u)$. Assume, in $\mathcal{P}_{\text{full}}$, u is input and (w, y) is output. Then, for every regularly implementable, autonomous $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}$, there exist a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$ through c, and a partition $u = (u_1, u_2)$ such that*

- *in $\mathcal{C}$, (y, u_2) is input and u_1 is output,*
- *in $\mathcal{K}_{\text{full}}$, u_2 is input and (w, y, u_1) is output.*

In general, the feedback transfer functions obtained in the above two theorems, are singular. In [15] it has been argued that many applications of control do not require the properness condition of the feedback transfer function and that the properness condition is, nevertheless, a very important special case.

6 Disturbances and regular interconnection

In section 3 we have formulated the problems of stabilization and pole placement for a given plant $\mathcal{P}_{\text{full}}$ with to-be-controlled variable w and control variable c. In most system models, an unknown external disturbance variable, d, also occurs. The stabilization problem is then to find a controller acting on c such that whenever $d(t) = 0$ $(t \geq 0)$, we have $w(t) \to 0$ $(t \to \infty)$. Typically, the disturbance d is assumed to be free, in the sense that *every* $\mathfrak{C}^\infty$ function d is compatible with the equations of the model. As an example, think of a model of a car suspension system given by $R_1(\frac{d}{dt})w + R_2(\frac{d}{dt})c + R_3(\frac{d}{dt})d = 0$, where d is the road profile as a function of time. In the stabilization problem, one puts $d = 0$ and solves the stabilization problem for the full plant $\mathcal{P}_{\text{full}}$ represented by $R_1(\frac{d}{dt})w + R_2(\frac{d}{dt})c = 0$. In doing this, one should make sure that the stabilizing controller $\mathcal{C}$: $C(\frac{d}{dt})c = 0$, when connected to the actual model, *does not put restrictions on d*. The notion of regular interconnection captures this, as explained below:

Consider the full plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}}$. An *extension* of $\mathcal{P}_{\text{full}}$ is a behavior $\mathcal{P}^{\text{ext}}_{\text{full}} \in \mathfrak{L}^{\mathtt{v}+\mathtt{c}+\mathtt{d}}$ (with $\mathtt{d}$ an arbitrary positive integer), with variables (w, c, d), such that

- d is free in $\mathcal{P}^{\text{ext}}_{\text{full}}$,
- $\mathcal{P}_{\text{full}} = \{(w, c) \mid \text{ such that } (w, c, 0) \in \mathcal{P}^{\text{ext}}_{\text{full}}\}$.

Thus, $\mathcal{P}^{\text{ext}}_{\text{full}}$ being an extension of $\mathcal{P}_{\text{full}}$ formalizes that $\mathcal{P}_{full}$ has exactly those signals (w, c) that are compatible with the disturbance $d = 0$ in $\mathcal{P}^{\text{ext}}_{\text{full}}$. Of course, a given full behavior $\mathcal{P}_{full}$ has many extensions.

For a given extension $\mathcal{P}^{\text{ext}}_{\text{full}}$ and a given controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, we define the extended controlled behavior by

$$\mathcal{K}^{\text{ext}}_{\text{full}} = \{(w, c, d) \mid (w, c, d) \in \mathcal{P}^{\text{ext}}_{\text{full}} \text{ and } c \in \mathcal{C}\}.$$

A controller $\mathcal{C}$ shall be acceptable only if the disturbance d remains free in $\mathcal{K}^{\text{ext}}_{\text{full}}$, *for any possible extension* $\mathcal{P}^{\text{ext}}_{\text{full}}$. It turns out that this is guaranteed exactly, by the regularity of the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$!

Theorem 9 : *The following statements are equivalent.*

- *The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular,*
- *for any extension $\mathcal{P}^{\text{ext}}_{\text{full}}$ of $\mathcal{P}_{\text{full}}$, d is free in $\mathcal{K}^{\text{ext}}_{\text{full}}$.*

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